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DIFFRACTION OF SHOCK WAVES AT SMALL ANGLES IN A PERFECT COMPACTIBLE MEDIUM

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A problem of diffraction of a plane shock wave on a wall forming a small angle with the vector normal to the front of the incident wave, is investigated. The wave propagates through a perfect compactible medium. The model of the medium is used to give an approximate description of mechanical behavior of porous materials subjected to pressures greatly exceeding their yield point, with the compressibility of the lattice skeleton remaining insignificant as compared with deformation of the filler, and for the time being we can ignore it.

The solution is obtained using the method given in [1] for solving the Lighthill's problem [2]. A method, in which the present problem is regarded as a problem of singular perturbations, is also used. The theory of approximate conformal mapping of neighboring regions [3] is used to obtain the first approximation in the outer expansion, with the smaller singularity situated at the corner point. When the degree of compaction is small (weak shock wave), then a uniform first approximation can be constructed using the method of matching asymptotic expansions [4]. The asymptotics of the solution near the concave angle coincides with the known asymptotics of a solution near the edge of a wedge [4]. The solution about a convex angle is divergent, and in this case a bounded solution must be sought utilizing a different scheme of flow, e. g. taking into account the free surface formed because of the flow separation.

When the deformation becomes considerable (strong shock wave), it is shown that the region of irregularity is finite, and the problem becomes more complex since even the first term of the outer expansion depends on the inner solution.

It should be noted that the present problem differs from the problem of irregular reflection where the asymptotics is influenced by the nearness of the triple point to the reflecting surface (the first steps in the study of this problem were based in [5] on the short wave theory). The asymptotic case considered here is the exact opposite; the triple point is situated very far from the reflecting surface.

1. Formulation of the problem. Let a perfect compactible medium occupy a region bounded by a rigid wall AOE, and let the ray OE form a small angle with the continuation of the ray AO (Fig. 1). The plane shock wave with constant parameters (p_0 is pressure, u_0 and d_0 are the mass velocity and wave velocity) behind the front moves in the direction parallel to AO and reaches, at the instant $t = 0$, the corner point of the wall O. The problem is that of computing a two-dimensional selfsimilar motion of the medium resulting from diffraction at $t > 0$.

We assume that the medium ahead the front is at rest and the pressure is zero; the density of the medium is $\rho = \rho_0$ ahead the front and $\rho = \rho_1 = \text{const} > \rho_0$ $(-\infty < t < \infty)$ behind the front in accordance with the model used. Let us denote by p' and $\mathbf{u}'$ the pressure and mass velocity vector in a fixed x, y -coordinate system (Fig. 1) and seek the equation of the front in the form

$$x = d_0 t\,[1 + f'(y, t)]$$

Let us introduce the dimensionless and selfsimilar variables, and the equation of the front surface

$$\xi = x/d_0 t, \quad \eta = y/d_0 t, \quad p = p'/p_0$$
$$\mathbf{u} = (u, v) = \mathbf{u}'/u, \quad \xi = \xi(\eta) = 1 + f(\eta)$$

Using the ξ, η -variables we can write the system of Euler equations in the region Ω occupied by the medium in motion, the boundary conditions of the problem

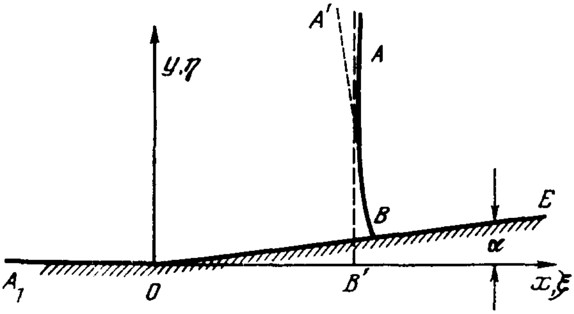

Fig. 1

at the front and at the wall, and the conditions of compatibility at the point B, in the form ($\mathbf{n}$ is the outward normal to the boundary)

$$\Omega : \operatorname{div}\mathbf{u} = 0, \quad K\mathbf{u} = \theta\nabla p \quad (p \to 1, \mathbf{u} \to (1, 0), r \to \infty) \tag{1.1}$$
$$AB : p = |\mathbf{u}|^2, \quad \mathbf{u} = \frac{\xi - \eta\xi'}{1 + \xi'^2}(1, -\xi') \quad (\xi(\eta) \to 1, \eta \to \infty)$$
$$A_1OB : \mathbf{u}\cdot\mathbf{n} = 0, \text{at the point } B : \eta/\xi = v/u = \operatorname{tg}\alpha$$
$$K = (\xi - \varepsilon u)\partial/\partial\xi + (\eta - \varepsilon v)\partial/\partial\eta, \quad \varepsilon = 1 - \theta = 1 - \rho_0/\rho_1$$
$$r^2 = \xi^2 + \eta^2, \quad \xi' = d\xi/d\eta$$

We shall seek the solution in the form of series in terms of small parameter α

$$p = 1 + \alpha p_1 + \alpha^2 p_2 + \ldots, \quad u = 1 + \alpha u_1 + \alpha^2 u_2 + \ldots$$
$$v = \alpha v_1 + \alpha^2 v_2 + \ldots, \quad f = \alpha f_1 + \alpha^2 f_2 + \ldots$$

Substituting these series into (1.1), we obtain the following problem for the first order approximation to which we shall restrict our search:

$$\Omega\colon \operatorname{div} \mathbf{u}_1 = 0,\ K_1\mathbf{u}_1 = \theta\nabla p_1\ (p_1 \to 0,\ \mathbf{u}_1 \to 0,\ r \to \infty) \tag{1.2}$$

$$AB\colon p_1 = 2u_1,\ u_1 = f_1 - \eta f_1',\ v_1 = -f_1'\ (f_1 \to 0,\ \eta \to \infty)$$

$$A_1OB\colon \partial p_1/\partial n = 0,\ \text{at the point } B\colon v_1 = 1,\ \eta = \alpha$$

$$(\mathbf{u}_1 = (u_1, v_1),\ K_1 = (\xi - \varepsilon)\partial/\partial\xi + \eta\partial/\partial\eta)$$

Eliminating $\mathbf{u}_1$ and f_1 from (1.2), we obtain

$$\Omega\colon \Delta p_1 = 0 \quad (p_1 \to 0,\ r \to \infty) \tag{1.3}$$

$$AB\colon 2\theta\eta\frac{\partial p_1}{\partial\xi} - (\eta^2 - \theta)\frac{\partial p_1}{\partial\eta} = 0,\ \int\limits_{BA}\frac{\partial p_1}{\eta} = -2 \tag{1.4}$$

$$A_1OB\colon \partial p_1/dn = 0 \tag{1.5}$$

where the second condition in (1.4) follows from the conditions prevailing at the front, and the conditions of compatibility (1.2). Thus in the first approximation, the problem reduces to that of determining a function p harmonic in Ω using the boundary conditions (1.4) and (1.5).

2. Solution of the problem under Lighthill's assumptions. Following of the method used in [1,2], we transfer the boundary conditions to the unperturbed boundary $A_1OB'A$ of the region Ω. The assumption of smallness of the perturbations everywhere in the region yields the following conditions $v_1 = 0$ on A_1O and $v_1 = 1$ on OB. In this case the condition (1.5) is replaced by [1,2] ($\delta(\xi)$ is the delta function)

$$AOB'\colon \partial p_1/\partial n = -\varepsilon\delta(\varepsilon)/\theta$$

Let us introduce the complex variable $\zeta = \xi + i\eta$ and the function $P = p_1 + iq$ analytic in Ω. The transformation $\zeta_1 = 1 - (1 - \zeta)^2 = \xi_1 + i\eta_1$ maps the region Ω conformally onto the upper half-plane ζ_1 with the normalizing conditions $\zeta_1(0) = 0,\ \zeta_1(1) = 1,\ \zeta_1(\infty) = \infty$. For the function $dP/d\zeta_1 = dp_1/d\xi_1 - i\partial p_1/\partial\eta_1$ analytic in the upper half-plane ζ_1 the condition on the real axis can be written in the form

$$a\frac{\partial p_1}{\partial\eta_1} + b\frac{\partial p_1}{\partial\xi_1} = c \tag{2.1}$$

$$a = 2\theta\sqrt{\xi_1 - 1},\quad b = \xi_1 - 1 - \theta \quad (\xi_1 > 1)$$

$$a = 1,\quad b = 0 \quad (\xi_1 < 1)$$

$$c = -\varepsilon\delta(\xi_1)/(2\theta) \quad (=\infty < \xi_1 < \infty)$$

The inhomogeneous Riemann — Hilbert problem with discontinuous coefficients stated above can be solved using the method given in [2,6]. The uniqueness of the solution is secured by the condition at infinity. Let us write the final result

$$\frac{dP}{d\zeta} = R(\zeta)\left[\frac{\varepsilon(1 + 3\theta)}{\pi\theta\zeta(\zeta - 2)} + A_0\right] \quad \left(R = \frac{1}{(1 - \zeta)^2 + 2\theta(1 - \zeta) + \theta}\right) \tag{2.2}$$

Here the constant A_0 is determined from the integral condition (1.4).
When the density change is small $(\varepsilon \ll 1,\ \theta \approx 1)$, we obtain

$$P \approx 4/[\pi(2-\zeta)] \quad (|\zeta| = O(1)) \tag{2.3}$$
$$P \approx -\frac{\varepsilon}{2\pi}\ln\zeta + \text{const} \quad (|\zeta| \to 0)$$

to within the terms of higher order of smallness.

From (2.2), (2.3) and (1.2) it follows that the pressure has a logarithmic singularity at the point $\zeta = 0$, and the velocity at the points $\zeta = 0$ and $\zeta = \varepsilon$. An analogous result was obtained in [1,2].

The present problem is a problem of singular perturbations. We find an uniformly usable first approximation by the method of matching asymptotic expansions. For the first term of the outer expansion we have the problem (1.2) which can be reduced to (1.3)—(1.5). Its solution is not unique, since the singularity appearing in the conditions (discontinuity) in the wall makes it possible to supplement the solution with a function possessing a singularity at the point O and exerting no influence on the boundary conditions. In choosing the solution we shall follow the general principles of minimizing the singular character and of ability to match with the additional expansion [4].

3. Solution of the problem by the matching of asymptotic expansions (first order approximation). Let $\zeta = \zeta(\zeta_1)$ be a function mapping the half-plane $\operatorname{Im}\zeta_1 > 0$ onto the perturbed region of flow Ω with the normalizing conditions $\zeta(0) = 0,\ \zeta(1) = \zeta_B,\ \zeta(\infty) = \infty$. On the real axis the condition for the function $dP/d\zeta_1$ analytic in the upper half-plane ζ_1 which follows from (1.4) and (1.5), is written in the form

$$a_1(\xi_1)\frac{\partial p_1}{\partial\eta_1} + b_1(\xi_1)\frac{\partial p_1}{\partial\xi_1} = 0 \tag{3.1}$$

$$a_1 = 1,\ b_1 = 0 \qquad (\xi_1 < 1)$$

$$a_1 = a_2\frac{\partial\xi}{\partial\xi_1} - b_2\frac{\partial\eta}{\partial\eta_1},\quad b_1 = a_2\frac{\partial\eta}{\partial\xi_1} + b_2\frac{\partial\xi}{\partial\eta_1} \quad (\xi_1 > 1)$$

$$a_2 = 2\theta\eta(\xi_1),\ b_2 = \theta - \eta^2(\xi_1)$$

The singularity $\partial p_1/\partial\eta_1$ at the point O does not appear here. It arises in Sect. 2 from the assumption that the expansion is regular, and this is not the case in the neighborhood of $\zeta = 0$.

We can write the uniform approximations to the function $\zeta(\zeta_1)$ and its derivative in the region $\operatorname{Im}\zeta_1 > 0$ in the form

$$\zeta \approx 1 + i\sqrt{\zeta_1 - 1},\quad \frac{d\zeta}{\partial\zeta_1} \approx \frac{i}{2\sqrt{\zeta_1 - 1}} \quad (|\zeta| > r_0) \tag{3.2}$$

$$\frac{d\zeta}{d\zeta_1} \approx \frac{ie^{i\pi\delta_1}}{2\zeta_1^{\delta}\sqrt{\zeta_1 - 1}} \quad (|\zeta| < r_0)$$

where r_0 is a number satisfying the conditions $r_0 \gtrsim \delta,\ \ r_0 \ll 1$ and the radical should be uniformized in such a manner that $\sqrt{\xi_1 - 1} > 0$ when $\xi_1 > 1,\ \delta_1 = \delta/(1-\delta),\ \delta = \alpha/\pi$.

We prove (3.2) using the method of approximate conformal mapping [5]. Let us consider the Christoffel – Schwarz transformation $\zeta = g(\zeta_2)$ of the half-plane $\operatorname{Im} \zeta_2 > 0$ onto the region $AOBA'$ (Fig. 1) with rectilinear boundaries

$$g(\zeta_2) = C_0 \int_0^{\zeta_2} \frac{dz}{z^{\delta}(z-1)^{1/2-\delta_2}}$$

with normalizing conditions $g(0) = 0$, $g(1) = \zeta_B$, $g(\infty) = \infty$ and the transformation $\zeta_3 = -1/\zeta_2$. The mapping of Ω in the plane $\zeta_3 = \xi_3 + i\eta_3$ is a region lying near the half-plane $\operatorname{Im} \zeta_3 > 0$ with the variation of the boundary $\eta_3 = \eta_3(\xi_3)$ different from zero on the interval $[-1, 0]$ (curve 1 in Fig. 2). Equality of the small angles α_1 and α_2 made by the curve $\eta_3(\xi_3)$ with the axis $\eta_3 = 0$ at the ends of the above interval, can be achieved by choice of δ_2. Consider the transformation $\zeta_3 = \zeta_3(\zeta_1)$ [5] of the upper half-plane ζ_1 onto the upper half-plane ζ_3 where one small segment of width 1/2 is removed and an identical one added on the interval $[-1, 0]$ and forming with the axis $\eta_3 = 0$ angles equal to α_1, on the region approximating the mapping of Ω (Fig. 2).

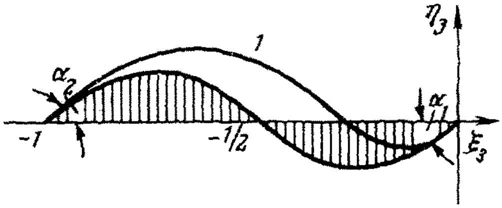

Fig. 2

Thus the mapping of Ω in the ζ_1-plane will be represented by a region which is near to the half-plane in the sense of Lavrent'ev [5] who gave the estimates of the mapping of this region onto the upper half-plane. Separating the main part of the mapping $\zeta = g\{\zeta_2[\zeta_3(\zeta_1)]\}$, we arrive at (3.2). The uniform estimate of the relative difference with respect to the derivatives is of the order of $\delta \ln \delta$, and that with respect to the functions, is of the order of δ.

Using (3.2) we obtain the following expressions for the coefficients a_1 and b_1 which appear in (3.1): $a_1(\xi_1) \approx a(\xi_1)$, $b_1(\xi_1) \approx b(\xi_1)$ where $a(\xi_1)$ and $b(\xi_1)$ are the same as in Sect. 2.

A solution of the Riemann – Hilbert problem (3.1) with discontinuous coefficients has the form

$$\frac{dP}{d\zeta_1} = \frac{1}{2} A\left[\sqrt{\zeta_1 - 1}\,(\zeta_1 - 1 - \theta + 2i\theta\sqrt{\zeta_1 - 1})\right]^{-1} \tag{3.3}$$

$$\frac{dP}{d\zeta} = \begin{cases} AR(\zeta) & (|\zeta| > r_0) \\ A\zeta^{\delta_1} e^{-i\pi\delta_1} R(\zeta) & (|\zeta| < r_0) \end{cases}$$

The uniqueness of the solution is ensured by the condition that $dP/d\zeta_1$ is integrable at infinity. The constant A is determined from the second condition of (1.4), and the function P is given by a single formula

$$P = A \int_{\infty}^{\zeta} \frac{dz}{(1-z)^2 + 2\theta(1-z) + \theta}$$

since the discrepancy in the expressions for $dP/d\zeta$ is not large.

When $\varepsilon \ll 1$, we have

$$P \approx 4/[\pi(2-\zeta)] \qquad (3.4)$$

to within the terms of the order of ε, and this coincides with the result (2.3) everywhere except in the small neighborhood of the point $\zeta = 0$. From (3.4) it follows that maximum pressure is obtained at the foot of the shock wave and is equal to $1 + 4\delta$. The complex velocity $w = u_1 - iv_1$ can be determined from the equation obtained from (1.2)

$$\frac{\partial w}{\partial r_1} = \frac{\theta}{r_1}\frac{dP}{d\zeta_0} \quad (r_1 = \sqrt{(\xi-\varepsilon)^2 + \eta^2},\ \zeta_0 = r_1 e^{i\beta_1} = \zeta - \varepsilon) \qquad (3.5)$$

This concludes the process of constructing the first term of the outer expansion. The solution is not regular in the neighborhood of the points $\zeta = 0$, ε. The singularity of the solution at the corner point of the wall is explained in physical terms. The singularity in the velocities is displaced by the unperturbed flow to the point $\zeta = \varepsilon$, and this is caused by the linearization of the equations.

The matching can be carried out only if the zone of strong perturbations is small. To determine the size of the region of irregularity, we substitute the principal part of the inner expansion of the outer solution into (1.1). The order of the discarded terms near $r_1 = 0$ is equal to $\varepsilon\delta^2 r_1^{-1} \ln r_1$, and that of the retained terms is equal to δ. The outer solution can be used as long as $r_1 / |\ln r_1| \gg \varepsilon\delta$. When $\varepsilon = O(1)$, $r_1 = O(1)$ provided that δ is of first order of smallness conpared with unity. This means that the linear dimension of the zone of strong perturbations is of the order of ε, i.e. of the order of the distance between the points $\zeta = 0$ and $\zeta = \varepsilon$ which between them embrace the irregularity. When $\varepsilon = O(1)$, the region of irregularity is finite, and the outer expansion depends on the inner expansion already in the first approximation. If on the other hand the region of irregularity is small, the first terms of the outer expansion is obtained independently, and the first term of the inner-expansion is obtained by matching with the inner expansion of the first term of the outer expansion [4].

Thus the validity of (3.3) is apparently restricted by the values $\varepsilon = o(1)$ (weak shock waves) since, as we said before, the solution may be different when $\varepsilon = O(1)$

Let us now restrict ourselves further, to the case $\varepsilon \ll r_0$ for which an uniform first order approximation is obtained in the simplest manner. We have $r_1 = r + O(\varepsilon)$ when $r \to r_0$. We construct in the neighborhood of $|\zeta| < r_0$ a solution of the system (1.1) satisfying the conditions of impermeability at the wall when $\beta = \alpha, \pi$ $(\zeta = re^{i\beta})$, and the asymptotics obtained from (3.3) and (3.5)

$$\frac{\partial W}{\partial r} \approx \delta_1 r^{\delta_1 - 1} e^{i\delta_1(\beta-\pi)}, \quad p_1 = \frac{2}{\pi} \quad (r \to r_0, \alpha < \beta < \pi, W = u - iv) \qquad (3.6)$$

Direct substitution confirms that the required velocity field is nearly potential when $r \to r_0$. We shall seek a solution under the assumption that the velocities are potential everywhere in the region indicated, and have the smallest possible singularity when $r \to 0$. We find that the solution is obtained by continuing the asymptotics (3.6) into the region

$$\frac{\partial W}{\partial r} = \delta_1 r^{\delta_1 - 1} e^{i\delta_1(\beta - \pi)} \quad (r < r_0, \alpha < \beta < \pi)$$

When $\delta_1 > 0$, the derivative $\partial W / \partial r$ is integrable and we obtain

$$W = r^{\delta_1} e^{i\delta_1(\beta - \pi)} \tag{3.7}$$

When $\delta_1 < 0$, we have $| W | \sim r^{\delta_1} \to \infty$ as $r \to 0$. In this case the function P_1 admits the representation

$$p_1 = 2 / \pi + O(\delta^{-1} \varepsilon r^{2\delta_1}) + O(r^{1+\delta_1})$$

and we also find that $p \to -\infty$ $(r \to 0,\ \delta_1 < 0)$.

Thus when $\alpha > 0$, the pressure deviates from a constant value by a small term and has a singularity when $\alpha < 0$. The character of the singularity indicates that the flow becomes separated from the wall and forms a free surface in the small neighborhood of the point $r = 0$. When $\alpha < 0$, the search for a bounded solution should take this free surface into account.

We note that the asymptotics (3.7) coincides with the asymptotics of the solution near the edge of a wedge when a perfect fluid flows past it [4].

In the region $| \zeta | > r_0$ the velocity w can be found by integrating (3.5). A check shows that the solutions for w with $r \to r_0$ from the inside and from the outside, are basically the same.

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